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Approximations of Fuzzy Systems

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Abstract

A fuzzy system can uniformly approximate any real continuous function on a compact domain to any degree of accuracy. Such results can be viewed as an existence of optimal fuzzy systems. Li-Xin Wang discussed a similar problem using Gaussian membership function and Stone-Weierstrass Theorem. He established that fuzzy systems, with product inference, centroid defuzzification and Gaussian functions are capable of approximating any real continuous function on a compact set to arbitrary accuracy. In this paper we study a similar approximation problem by using exponential membership functions

Keywords: *fuzzy systems, fuzzy approximation, centriod defuzzification*

1. Introduction

Fuzzy systems theory has number of applications in various fields namely control systems, signal processing, image processing etc. The main objective of these applications is to construct a fuzzy system to approximate the desired control or decision. Mathematically this is in fact to find a mapping from the input space to the output space which can approximate the desired function within a given accuracy. Hence the problems of designing fuzzy systems can be viewed as approximation problems. Fuzzy system experts studied several approximation problems of fuzzy systems that are closely related with fuzzy basic functions. Recently Li-Xin Wang discussed a fuzzy system approximation problem using Gaussian membership function. He established that fuzzy systems with product inference, centroid defuzzification and Gaussian functions are capable of approximating any real continuous function on a compact set to arbitrary accuracy. In this paper we study a similar approximation problem by using simple exponential membership functions.

2. Fuzzy Systems

A universe of discourse U is a collection of objects which can be discrete or continuous. A fuzzy set A in a universe of discourse U is characterized by a membership function $\mu_A:U\rightarrow[0, 1]$. For example if U is the set of all human beings, the concepts such as 'child', 'young', 'old' and 'very old' are called fuzzy concepts that are characterized by fuzzy sets. The fuzzy sets are further characterized by the following four principal elements.

1. Fuzzification Interface
2. Fuzzy Rule Base
3. Fuzzy Inference Machine
4. Defuzzification Interface

These four principal elements constitute a fuzzy system in U. The architecture of a fuzzy system is given in Figure 1. Throughout this paper $U\subset R^n$ where R is the set of all real numbers. Since a multi-output system can be separated into a collection of single output systems, we shall consider multi-input-single output (MISO) fuzzy systems $f:U\subset R^n \rightarrow R$.

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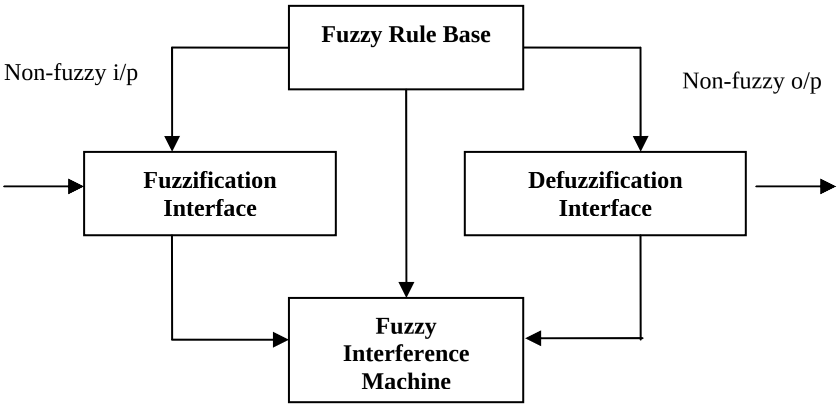

Figure 1. Architecture of Fuzzy Systems

3. Fuzzification Interface

The mapping of the observed input universe of discourse $U \subset R^n$ to the fuzzy sets defined in U is termed as fuzzification interface. Let (A_i, μA_i), be a fuzzy set defined in U for i=1,2,3….n. Let $x \in U$ be an input to the fuzzification interface. Then, the outputs of the fuzzification interface are $\mu A_i(x_i)$ for i=1,2,3….n . The following are the important factors that are used to determine a fuzzification interface.
(1) The number of fuzzy sets defined in the input universe of discourse and
(2) the specific membership functions.
We can view these two factors as design parameters of a fuzzification interface. Specifically, for a MISO fuzzy system, the design parameters of a fuzzification interface are (1) m_i, i=1,2…..n, the number of fuzzy sets defined in the subspace $\{x_i : x = (x_1,x_2,\ldots x_i,..,x_n) \in U\}$ of U corresponding to the i^{th} coordinates and (2) $\mu A_{(i,\ j)}$ for i=1,2,..,n, j=1,2,..,m_i, the membership function of the j^{th} fuzzy set defined in the i^{th} subspace of U.

4. Fuzzy Rule Base

The fuzzy rule base is a set of predefined linguistic labels in the form of “IF a set of conditions are satisfied, THEN a set of consequences are inferred”, where the conditions and the consequence are associated with fuzzy concepts.
For example, in the case of an n-input –single output fuzzy system, the fuzzy rule base may consist of the following rules:

Ri : IF x_1 is $A_{(1,\ j)}$ and x_2 is $A_{(2,\ j)}$ and …, x_n is $A_{(n,\ j)}$, THEN z is B^j (1)

Where x_1 ,x_2, …,x_n are the inputs to the fuzzy system, z is the output of the fuzzy system, $A_{(i,\ j)}$ and B^j , for j=1,2,…,K are linguistic terms and K is the number of fuzzy rules in the fuzzy rule base. By relating each linguistic term in the fuzzy rules with a membership function, we specify the meaning of the fuzzy rules in determined fuzzy sense. There are many different kinds of fuzzy rules; see [4] for a complete discussion.

5. Fuzzy Inference Machine

The fuzzy inference machine is a decision making logic which employs fuzzy rules from the fuzzy rule based determine fuzzy outputs of a fuzzy system corresponding to the fuzzified inputs to the fuzzy system. It is the fuzzy inference machine that simulates a human decision making procedure based on fuzzy concepts and linguistic statements. There are many different kinds of fuzzy logic which may be used in a fuzzy inference machine; see [4] for a comprehensive review.

6. Defuzzification Interface

The defuzzification interface defuzzifies the fuzzy output of a fuzzy system to generate non-fuzzy output. There are three existing defuzzification methods, namely;
centroid, max-criterion and mean of maximum (see [4] for details). The design parameters of a defuzzification interface are:
(1) Number of fuzzy sets defined in the output universe of discourse R;
(2) Specific membership functions of these fuzzy sets; and,
(3) Which defuzzification method is used.

In summary, a fuzzy system has the following design parameters
(P_1) Number of fuzzy sets defined in the input and output universes of discourse;
(P_2) Membership functions of these fuzzy sets;
(P_3) Number of fuzzy rules in the fuzzy rule base;
(P_4) Linguistic statements of the fuzzy rules;
(P_5) Decision making logic used in the fuzzy inference machine; and,
(P_6) Defuzzification method.

The number of fuzzy sets defined in the input and output universes of discourse and the number of fuzzy rules in the fuzzy rule base heavily influence the complexity of a fuzzy system, where complexity includes time complexity and space complexity. These parameters can be viewed as structure parameters of a fuzzy system. In general, the larger these parameters are, the more complex is the fuzzy system, and the higher is the expected performance of the fuzzy system. Hence, there is always a trade off between complexity and accuracy in the choice of these parameters; and their choice is usually quite subjective.

The membership functions of the fuzzy sets heavily influence the "smoothness" of the input- output surface determined by fuzzy system. In general, the "sharper" the membership functions are, the less smooth is the input-output surface. The choice of membership functions is also quite subjective. The linguistic statements of the fuzzy rules are the heart of a fuzzy system in the sense that it is these linguistic statements assists in the effective representation and use of the information. The fuzzy rules usually come from two sources; the experts, and training data. A general method to generate fuzzy rules from numerical data was proposed in [8], and was successfully applied to temperature prediction, truck backer–upper control, and chaotic time-series prediction. Another method was proposed in [2] to generate fuzzy rules from numerical data using vector quantization.

The decision making logic used by fuzzy inference machine is very important, and may be the most flexible component in the fuzzy system. A sophisticated fuzzy system may need a sophisticated fuzzy inference machine. The role of the defuzzification strategy in a fuzzy system is somewhat unclear because there are only three defuzzification methods available, among which the centroid method seems to provide the best performance for most application [4].

Let Z be a set of real continuous functions on U. Then Z is an algebra if Z is closed under addition, multiplication and scalar multiplication. Z separates points on U if for every x, $y \in U$, $x \neq y$, there exists $f \in Z$ such that $f(x) \neq f(y)$. *Z* vanishes at no point of U if for each $x \in U$ there exists $f \in Z$ such that $f(x) \neq 0$. We use the following theorem due to Stone and Weierstrass [6].

Theorem 1. Let Z be a set of real continuous functions on compact set U. Suppose Z is an algebra, Z separates points on U and *Z* vanishes at no point of U. Then, the uniform closure of Z consists of all real continuous functions on U, that is (Z, d_∞) is dense $(C[U], d_\infty)$ where $d_\infty(f_1, f_2) = \text{Sup}\{|f_1(x) - f_2(x)| : x \in U\}$.

7. Fuzzy Systems are Universal Approximators

We know that MISO fuzzy systems can be viewed as mappings from a non-fuzzy universe of discourse $U \subset R^n \rightarrow R$, and these mappings are characterized by the six design parameters P1– P6. All these mappings consist of a function space X. Since there are great flexibilities in choosing the design parameters, X is a very large space. In this section, we analyse the properties of a subset of X which is determined by fixing some design parameters in the following way:
(a) Fuzzy rules in the fuzzy rules base are all in the form of (1);
(b) All membership functions are of the following exponential form:

$$\mu A_{(i,\,j)}(x_i) = \frac{1}{x^*_{(i,j)}} \exp\left(\frac{-x_i}{x^*_{(i,j)}}\right) \tag{2}$$

Where i=0,1,2,…n, j = 1,2,…,K(K is the number of fuzzy rules in the fuzzy rule base), i = 0 represents the membership functions for the output space, and i =1,2,…,n represent the membership functions for the input space; $A_{(i,\,j)}$ has the same meaning as in(1) for i = 1,2,..,n; $A_{(0,\,j)} = B^j$; $x^*_{(i,j)}$ is the point in the i^{th} input subspace for i = 1,2,..,n and $x^*_{(0,j)}$ is in the output space at which the fuzzy set ($A_{(i,j)}$, $\mu A_{(i,\,j)}$) achieves its maximum membership value; and characterizes the shape of the exponential membership function;

(c) Product inference logic [3] (which is specified below) is used in the fuzzy inference machine; and,

(d) Centroid defuzzification method(which is also specified below)is used in the defuzzification interface.

Definition: The set of fuzzy systems with product inference, centroid defuzzification and exponential membership functions, denoted by Y in the sequel, consists of all functions.

$$f{:}U\subset R^n \rightarrow R \text{ defined by } f(x) = \frac{\sum_{j=1}^{K}(z_j^*\prod_{i=1}^{n}\mu A_{(i,j)}(x_i))}{\sum_{j=1}^{K}(\prod_{i=1}^{n}\mu A_{(i,j)}(x_i))}, \tag{3}$$

x = (x_1, x_2,.., x_n)∈U where K is the number of fuzzy rules in the fuzzy rule base, $\mu A_{(i,\,j)}(x_i)$ is the exponential membership function in (2) and z_j^* is the point in the output space R at which μB^j achieves its maximum value. We assume K ≥ 1, and that U is compact [5].

From Equation (3), we observe that if we view the fuzzy inference machine and defuzzification interface as an integrated part, then product inference logic can be explained as that the 'weight' of Rule j to the contribution of determining the output of the fuzzy system for input x equals $\prod_{i=1}^{n}\mu A_{(i,\,j)}(x_i)$. Centroid defuzzification means that the non-fuzzy output of the fuzzy system is a weighted sum of the K points in R at which the membership function characterizing the linguistics terms in the conclusion parts of the K rules achieve their maximum values, where the 'weights' are determined by the product inference machine.

The design parameters of the fuzzy systems in Y are:

(i) m_i, *i* =1,2,..n, the number of fuzzy sets defined in the i^{th} subspace of the input universe of discourse U, and , m_0, the number of fuzzy sets defined in the output space R;

(ii) $x^*_{(i,j)}$ and z_j^* (i =1,2,…,n, j =1,2,…,K), the parameters of the exponential membership functions of the fuzzy sets defined in the input and output spaces;

(iii) K, the number of fuzzy rules in the fuzzy rule base, with K ≥ 1; and,

(iv) The specific statements of the fuzzy rules which are in the form of (1).

Let d_∞ (f_1, f_2) be the sup-metric [5] defined by $d_\infty(f_1, f_2) = \text{Sup}\{|f_1(x) - f_2(x)| : x\in U\}$. Then (Y, d_∞) is a metric space.

Proposition 1. Y is non-empty.

Proof. Follows from the assumption K ≥ 1.

Proposition 2. (Y, d_∞) is well-defined.

Proof: Since Y is non-empty, we only need to prove that the denominators of (3) is nonzero for any x∈U. Based on (2), the exponential membership functions are nonzero; hence, the denominator of (3) is nonzero.

From the proof of Proposition 2 we see that if we change the membership functions into the triangular form, then the resulting d_∞ may not be well-defined, because for an arbitrary f in such Y we cannot guarantee that the denominator of f is non zero for every x ∈U.

Next we use Theorem 1 to prove that (Y, d_∞) is dense in (C[U], d_∞), where C[U] is the set of all real continuous functions defined on the compact set U.
In order to use Theorem 1, we need to show that Y is an algebra, Y separates points on U, and, Y vanishes at no point of U.

Proposition 3. (Y, d_∞) is algebra.

Proof:

Let $f_1, f_2 \in$ Y. Then $f_1(x) = \dfrac{\sum_{j=1}^{K1} (z1^*_j \prod_{i=1}^{n} \mu A1_{(i,j)}(x_i))}{\sum_{j=1}^{K1} (\prod_{i=1}^{n} \mu A1_{(i,j)}(x_i))}$

$$f_2(x) = \frac{\sum_{j=1}^{K2} (z2^*_j \prod_{i=1}^{n} \mu A2_{(i,j)}(x_i))}{\sum_{j=1}^{K2} (\prod_{i=1}^{n} \mu A2_{(i,j)}(x_i))}$$

$$f_1(x) + f_2(x) = \frac{\sum_{j1=1}^{K2}\sum_{j2=1}^{K2} (z1^*_{j1} + z2^*_{j2})(\prod_{i=1}^{n} \mu A1_{(i,j1)}(x_i)\mu A2_{(i,j2)}(x_i))}{\sum_{j1=1}^{K2}\sum_{j2=1}^{K2} (\prod_{i=1}^{n} \mu A1_{(i,j1)}(x_i)\mu A2_{(i,j2)}(x_i)))}$$

$$f_1(x)f_2(x) = \frac{\sum_{j1=1}^{K2}\sum_{j2=1}^{K2} (z1^*_{j1}. z2^*_{j2})(\prod_{i=1}^{n} \mu A1_{(i,j1)}(x_i)\mu A2_{(i,j2)}(x_i))}{\sum_{j1=1}^{K2}\sum_{j2=1}^{K2} (\prod_{i=1}^{n} \mu A1_{(i,j1)}(x_i)\mu A2_{(i,j2)}(x_i)))}$$

$$cf_1(x) = \frac{\sum_{j=1}^{K1} (cz1^*_j \prod_{i=1}^{n} \mu A1_{(i,j)}(x_i))}{\sum_{j=1}^{K1} (\prod_{i=1}^{n} \mu A1_{(i,j)}(x_i))}$$

□

Proposition 4. (Y, d_∞) separates points on U.

Proof : We prove this by constructing a required f. Let $x^0 \neq y^0$. We wish to construct f such that $f(x^0) \neq f(y^0)$. Let $x^0 = \left(x_1^0, x_2^0, ...x_n^0\right)$ and $y^0 = \left(y_1^0, y_2^0, ...y_n^0\right)$. If $x_i^0 \neq y_i^0$, we define two fuzzy sets, $(A_{(i,1)}, \mu A_{(i,1)})$ and $(A_{(i,2)}, \mu A_{(i,2)})$ in the ith subspace of U, with:

$$\mu A_{(i,1)}(x_i) = \frac{1}{x_i^0}\exp\left(\frac{-x_i}{x_i^0}\right) \text{ and } \mu A_{(i,2)}(x_i) = \frac{1}{y_i^0}\exp\left(\frac{-x_i}{y_i^0}\right). \quad (4)$$

If $x_i^0 = y_i^0$, then $A_{(i,1)} = A_{(i,2)}$ and $\mu A_{(i,1)} = \mu A_{(i,2)}$. Therefore, only one fuzzy set is defined in the ith subspace of U. We define two fuzzy sets, (B^1, μB^1) and (B^2, μB^2), in the output universe of discourse R, with $\mu B^j(z) = \frac{1}{z_j^*}\exp\left(\frac{-z}{z_j^*}\right)$ where j = 1,2 and z_j^* will be specified later. We choose two fuzzy rules in the form of (1) for the fuzzy rule base (i.e., K = 2). Now we have specified all the design parameters except z_j^* (j = 1,2), that is we have already obtained a function f which is in the form of (3) with K = 2 and $\mu A_{(i,j)}$, given by (4). With this f, we have:

$$f(x^0) = \frac{z_1^* \prod_{i=1}^{n} \mu A_{(i,1)}(x_i^0) + z_2^* \prod_{i=1}^{n} \mu A_{(i,2)}(x_i^0)}{\prod_{i=1}^{n} \mu A_{(i,1)}(x_i^0) + \prod_{i=1}^{n} \mu A_{(i,2)}(x_i^0)}$$

$$= \frac{z_1^* \prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{x_i}{x_i^0}) + z_2^* \prod_{i=1}^{n} \frac{1}{y_i^0} \exp(-\frac{x_i}{y_i^0})}{\prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{x_i}{x_i^0}) + \prod_{i=1}^{n} \frac{1}{y_i^0} \exp(-\frac{x_i}{y_i^0})}$$

$$= \frac{z_1^* \left[\frac{1}{x_1^0 . x_2^0 .... x_n^0} e^{-1}\right] + z_2^* \left[\frac{1}{y_i^0} \exp(-\frac{x_i}{y_i^0})\right]}{\left[\frac{1}{x_1^0 . x_2^0 .... x_n^0} e^{-1}\right] + \prod_{i=1}^{n} \left[\frac{1}{y_i^0} \exp(-\frac{x_i}{y_i^0})\right]}$$

Let $\left[\frac{1}{x_1^0 . x_2^0 .... x_n^0} e^{-1}\right] = \alpha$ and $\prod_{i=1}^{n} \left[\frac{1}{y_i^0} \exp(-\frac{x_i^0}{y_i^0})\right] = \beta$

Then $f(x^0) = \frac{z_1^* \alpha + z_2^* \beta}{\alpha + \beta} = \left(\frac{\alpha}{\alpha+\beta}\right) z_1^* + \left(\frac{\beta}{\alpha+\beta}\right) z_2^*$

Similarly $f(y^0) = \frac{z_1^* \prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{y_i^0}{x_i^0}) + z_2^* \prod_{i=1}^{n} \frac{1}{y_i^0} \exp(-\frac{y_i^0}{y_i^0})}{\prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{y_i^0}{x_i^0}) + \prod_{i=1}^{n} \frac{1}{y_i^0} \exp(-\frac{y_i^0}{y_i^0})}$

$$= \frac{z_1^* \prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{y_i^0}{x_i^0}) + z_2^* \left(\frac{e^{-1}}{y_1^0 . y_2^0 ... y_n^0}\right)}{\prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{y_i^0}{x_i^0}) + \left(\frac{e^{-1}}{y_1^0 . y_2^0 ... y_n^0}\right)}$$

Let $\left(\frac{e^{-1}}{y_1^0 . y_2^0 ... y_n^0}\right) = \gamma$ and $\prod_{i=1}^{n} \frac{1}{x_i^0} \exp(-\frac{y_i^0}{x_i^0}) = \delta$.

$f(y^0) = \frac{z_1^* \delta + z_2^* \gamma}{\delta + \gamma} = \left(\frac{\delta}{\delta+\gamma}\right) z_1^* + \left(\frac{\gamma}{\delta+\gamma}\right) z_2^*$. We need to prove that $f(x^0) \neq f(y^0)$, that is

$$\left(\frac{\alpha}{\alpha+\beta}\right) z_1^* + \left(\frac{\beta}{\alpha+\beta}\right) z_2^* \neq \left(\frac{\delta}{\delta+\gamma}\right) z_1^* + \left(\frac{\gamma}{\delta+\gamma}\right) z_2^*.$$

<u>Claim: $f(x^0) \neq f(y^0)$</u>

$f(x^0) = f(y^0) \Leftrightarrow \left(\frac{\alpha}{\alpha+\beta}\right) = \left(\frac{\delta}{\delta+\gamma}\right)$ and $\left(\frac{\beta}{\alpha+\beta}\right) = \left(\frac{\gamma}{\delta+\gamma}\right)$

$\Leftrightarrow \alpha(\delta+\gamma) = \delta(\alpha+\beta)$ and $\beta(\delta+\gamma) = \gamma(\alpha+\beta)$

$\Leftrightarrow \alpha\delta+\alpha\gamma = \delta\alpha+\delta\beta$ and $\beta\delta+\beta\gamma = \gamma\alpha+\gamma\beta$

$\Leftrightarrow \alpha\gamma = \delta\beta$ and $\beta\delta = \gamma\alpha$

$$\Leftrightarrow \left(\frac{1}{x_1^0 . x_2^0 .... x_n^0} e^{-1}\right)\left(\frac{e^{-1}}{y_1^0 . y_2^0 ... y_n^0}\right) = \prod_{i=1}^{n} \left[\frac{1}{y_i^0} \exp(-\frac{x_i^0}{y_i^0})\right] \prod_{i=1}^{n} \left[\frac{1}{x_i^0} \exp(-\frac{y_i^0}{x_i^0})\right].$$

$$\Leftrightarrow e^{-2} = \prod_{i=1}^{n} \exp(-\frac{x_i^0}{y_i^0}) \exp(-\frac{y_i^0}{x_i^0}) = \prod_{i=1}^{n} \exp(-\frac{(x_i^0)^2 + (y_i^0)^2}{x_i^0 y_i^0}).$$

$$\Leftrightarrow e^{-2} = \exp(-\sum_{i=1}^{n} \frac{(x_i^0)^2 + (y_i^0)^2}{x_i^0 y_i^0}).$$

$\Leftrightarrow 2 = \sum_{i=1}^{n} \frac{(x_i^0)^2 + (y_i^0)^2}{x_i^0 y_i^0}$. This is not possible as $x^0 \neq y^0$.

Hence $f(x^0) \neq f(y^0)$.

□

Proposition 5. (Y, d_∞) vanishes at no point on U.

Proof: By observing (3) and (2), we simply choose all $z_j^* > 0$ (j =1,2,..K), i.e., any $f \in Y$ with $z_j^* > 0$ serves as the required f.

□

The next theorem shows that the fuzzy systems in Y can approximate continuous functions.

Theorem 2. For any given real continuous function g on the compact set $U \subset R^n$ and arbitrary $\in > 0$, there exists $f \in Y$ such that $\mathrm{Sup}\{ |f(x)-g(x)| : x \in U \} < \varepsilon$

Proof: **I**t is obvious that Y is a set of real continuous function on U. The proof follows Theorem 1 and Propositions 3-5.

□

8. Conclusion

Thus a suitable collection of membership functions can be found for a fuzzy system that can be an optimal fuzzy system for a variety of problems.

References

[1] Hornik K, Stinchcombe M and White H. Multilayer feedforward networks are universal approximators. *Neural Networks*. 1989; 2: 359-366.

[2] Kosko B. Neural Networks and Fuzzy Systems. *Prentice-Hall Englewood Cliffs N J*. 1991.

[3] Larsen PM. Industrial Applications of fuzzy logic control. *Int J Man Mach Studies*. 1980; 12(1): 3-10.

[4] Lee CC. Fuzzy logic in control Systems: fuzzy logic controller, part II. *IEEE Trans on Syst Man and Cybern*. 1990; 20(2): 429-435.

[5] Naylor AW, Sell GR. Linear Operator Theory in Engineering and Science. *Springer-Verlag, Newyork.* 1982.

[6] Rudin W. Principles of Mathematical Analysis. *McGraw Hill, Inc*. 1976.

[7] Togai M, Watanabe H. Expert system on a chip: an engine for real-time approximate reasoning. *IEEE Expert System Mag*. 1986; 1: 55-62.

[8] Wang LX, Mendel JM. *Generating fuzzy rules by learning from examples*. Proc 6th IEEE International Symposium on Intelligent Control. *Washington D C*. 1991; 263-268.