

# A Pseudo-spectral Method with Least Squares Approach and Monomial Basis Functions for Fractional Ordinary Differential Equations

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## ABSTRACT

The present work combines the pseudo-spectral method with least square technique to approximate non-linear and linear differential equations that are bounded and it makes use of monomials as trial functions. The pseudo-spectral method involves the approximation of solutions to differential equations by taking the sum of truncated basis functions, and this feature differentiates this method from difference-based methods. The idea of the least square technique is to optimally minimize the sum of squares of the residual function of the differential equation. The Caputo fractional differential operator is generally used given that it allows for the inclusion of the conditions at the initial state and the boundaries in formulating the equation(s). Non-linear fractional differential equations are linearized using the quasilinearization approach, which is based on the Newton-Raphson method. The resulting linear algebraic equations are regularized using the Tikhonov regularization technique due to the density of the equations. Some problems are solved to demonstrate the efficiency of the proposed method and display its convergence and accuracy, respectively, of approximate solutions obtained. Comparison of the numerical solutions and exact solutions (where obtainable) is conducted, and the numerical results are further presented in tabular form and graphically. The results obtained show that the proposed method is computationally efficient. The process of regularizing the numerical schemes of non-linear fractional differential equations was also shown to give more accurate solutions.

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## 1. INTRODUCTION

Fractional calculus applications have inspired the development and study of numerical methods intended to solve fractional differential equations. Following Garrappa [1], the importance of numerical methods cannot be understated due to the complexities surrounding obtaining exact solutions to most nonlinear fractional differential equations. In some of the few instances where analytical solutions have been obtained, it

has required the introduction of unique analytical methods and led to interesting results. Kaplan and Bekir [2] introduced a new method that produced analytical solutions to time-fractional differential equations. By modifying the Riemann-Liouville derivative, the exponential function method was proposed to obtain rational function solutions for fractional differential equations that contain trigonometric and hyperbolic properties. The Sharma-Tasso-Olver time-fractional problem and the  $(3 + 1)$ -dimensional Korteweg-de Vries-Zakharov-Kuznetsov (KdV-ZK) time-fractional problem were solved to validate the method. New exact solutions were successfully obtained as hyperbolic, trigonometric and rational functions. Furthermore, Ghorbani [3] proposed an approach that obtained exact solutions to non-linear fractional differential equations. Unlike the variational iteration method, the proposed fractional iteration method [3] does not require a Lagrange multiplier to solve non-linear fractional differential equations and the solution is obtained in a direct process. The results revealed that the fractional iterative method was gave accurate solutions. A method that gives exact solutions to the fractional model vibration equation was introduced by Srivastava *et al.* [4]. The technique was based on the q-homotopy analysis transformation technique to investigate a fractional model of the vibration equation for large membranes [4]. The study also used the Laplace decomposition method to obtain solution to the fractional-order vibration equation.

Following Smith [5], numerical methods approximate solutions to mathematical problems and are easy to interpret, unlike analytical methods, because analytical methods can be quite tedious or impossible to implement on some mathematical problems. In a novel study, Li [6] introduced some numerical methods for solving fractional differential equations. The study introduced highly accurate numerical schemes for typical fractional differential equations using the parametric quintic spline method [6], and solved generalized fractional problems using the non-uniform mesh method. It was noticed that the attained results had higher order of accuracy than both the compact finite difference method with a fourth-order accuracy, and the central finite difference with a second-order accuracy with the computational cost in applying the method being acceptable. The theoretical analysis conducted revealed that the numerical methods proposed worked efficiently with a larger class of fractional differential equations. As observed in Li [6], the numerical scheme was seen to be highly stable and convergent for generalized fractional differential equations. Elzaki and Chamekh [7] introduced a novel method that applied decomposition when solving non-linear fractional initial value equations. The strategy was based on modifying the Adomian decomposition method, and it provided semi-analytical solutions to non-linear fractional differential equations [7]. Furthermore, Dehghan *et al.* [8] solved a system of non-linear fractional partial differential equations using the homotopy analysis method, and they discovered that in the application of the method, the method proved to be highly efficient, and gave highly accurate approximate solutions.

Li and Zeng [9] modified the finite difference method to solve fractional differential equations. The primary aim of Li and Zeng [9] was to study the stability and convergence of the fractional Euler method, the fractional Adams method, and a higher-order method based on the quadrature formula. The results showed that the three proposed methods are stable and convergent. Herrera *et al.* [10] presented the analysis and numerical simulation of the time-fractional diffusion equation using the mimetic finite difference method. The study was devoted to the numerical treatment of time-fractional diffusion equations with mixed boundary conditions. A new scheme based on the combination of the implicit finite difference method for the Caputo derivative in time and the mimetic finite difference method in space was considered in the study. It was concluded that the error between the exact and approximate solution was much better for the mimetic finite difference method. Zhao *et al.* [11] developed an adaptive finite element method for fractional differential equations using hierarchical matrices. The Riesz derivative was applied to find non-smooth solutions. The study showed that the uniform grids result in a sub-linear convergence rate while the proposed method yields optimal second-order accuracy. The study was based on the utilization of the hierarchical matrices ( $\mathcal{H}$  matrices) to present the stiffness matrix resulting from the finite element discretization of the fractional differential equations. A high-resolution two-mesh finite element algorithm for a two-dimensional space fractional Gray-Scott model was studied by Lui *et al.* [12]. The method included using an iterative method for solving the coupled non-linear system on a coarse grid, an interpolation formula to obtain coarse values on the fine time mesh and a linear finite element system on the temporal mesh constructed using two variables Taylor's formula. Theoretical results proved the stability and convergence of the fully discrete scheme.

Spectral methods are a class of techniques used in applied mathematics to solve differential equations numerically. These methods aim to write the solution of the differential equation as a linear combination of basis functions. For example, the Fourier spectral method assumes that the solution of the differential equation

is a linear sum of sinusoids [13]. Due to the high computational cost associated with numerically solving fractional differential equations, spectral methods are often considered and applied to solve fractional differential equations. The choice of basis functions is one feature that distinguishes spectral methods from finite difference or element methods. The basis functions in spectral methods are infinitely differentiable global functions [14]. A geometrically convergent pseudo-spectral method for multi-dimensional two-sided space fractional partial differential equations was presented by Oloniju et al. [15]. In addition to establishing the proposed scheme's accuracy, the scheme's convergence was shown to be geometrical, like other known variations of spectral-based methods. Costa [16] presented a spectral method for partial differential equations. The study showed that using the spectral-based spatial discrete differentiation operator makes the temporal integration formula work to a higher precision level with a very small step size. Baleanu et al. [17] introduced a Chebyshev-spectral method with a differential operational matrix for fractional differential equations involving the non-singular Mittag-Leffler kernel. Numerical examples showed the efficiency and accuracy of the proposed method. Oloniju et al. [18] described a pseudo-spectral method for time-distributed order two-sided space fractional differential equations. The first-kind shifted Chebyshev polynomials interpolated on the Gauss-Lobatto quadrature were used. The numerical results presented showed that the method is computationally efficient and accurate.

A numerical approximation of a two-term time-fractional differential equation using the Lagrange polynomial pseudo-spectral method was implemented by Shokri and Mirzaei [19]. The study used the Mittag-Leffler function for time integration to protect the high accuracy of the spectral approximation. The results in Shokri and Mirzaei [19] confirmed a spectral-based method's spectral convergence and efficiency.

A fundamental task in scientific computing is to estimate parameters in mathematical models from observations that are subject to errors. A common practice is to reduce the influence of errors by using the least square method [20]. The study in Mortari [21] highlighted how to obtain least squares solutions to initial value problems, boundary value problems and multi-value problems for non-homogeneous linear differential equations with non-constant coefficients of any order. The numerical approximation of fractional integrodifferential equations by the least square method and shifted Chebyshev polynomials was investigated by Mohammed [22]. The technique proposed in Mohammed [22] resulted in a good approximation; hence, the method was effective and had a high convergency rate. Numerical studies for solving a system of linear fractional integrodifferential equations by the least square method and shifted Chebyshev polynomials were carried out by Mahdy and Mohamed [23]. The proposed method reduced the system of linear fractional integrodifferential equations to a system of linear algebraic equations. The numerical results indicate that the approach is easy to implement and accurate. Zhang et al. [24] proposed the least-square residual power series method for the time-fractional differential equations. The method hybridizes the residual power series method and the least-square method. It was shown that the least-square residual power series method has an accelerated convergence rate compared to the classic residual power series method. Jia et al. [25] investigated an efficient technique based on the least-square method for fractional integrodifferential equations. The solvability and stability of the proposed numerical schemes were proven, and the technique obtained accurate approximate solutions.

Regularization techniques are commonly employed to address large condition numbers and ill-posed systems [26, 27]. Among these, Tikhonov regularization is one of the most widely used approaches for stabilizing numerical solutions of ill-posed problems and obtaining smooth approximations [28]. Several studies, including that of Calvetti and Reichel [29], have demonstrated the effectiveness of Tikhonov regularization in handling large-scale problems. In the present work, Tikhonov regularization is utilized to improve the stability of the resulting algebraic systems, particularly in the nonlinear case where ill-conditioning becomes more pronounced.

Recent developments in numerical methods for fractional differential equations have focused on hybrid spectral and wavelet-based approaches, particularly for nonlinear problems. For instance, Haar- and Hermite-wavelet-based methods have been successfully applied to nonlinear fractional differential equations, demonstrating improved numerical stability and accuracy [30, 31]. In addition, orthogonal polynomial-based approaches, such as those employing Chebyshev and Legendre bases, are widely used due to their favorable conditioning properties [32]. These advances highlight the importance of addressing conditioning issues in spectral approximations and further motivate the use of regularization techniques in the present work.

In this study, the numerical method chosen is the pseudo-spectral method. In contrast to finite difference methods, pseudo-spectral methods globally approximate a function and its derivatives. The main advantage of pseudo-spectral methods is their exponential or spectral convergence rate. It is also possible to achieve good accuracy with a relatively coarse grid using the pseudo-spectral methods [33]. In this study, we intro-

duce a method to solve linear and non-linear boundary value fractional differential equations by combining the pseudo-spectral method with least squares and discretizing using the monomial basis functions. For linear fractional differential equations, the resulting algebraic systems can be solved directly without the need for regularization, as stable and accurate solutions are obtained for the chosen approximation sizes. However, for nonlinear problems, the resulting linearized algebraic systems are dense and may be ill-conditioned; therefore, Tikhonov regularization is used to improve numerical stability. The novelty of this study lies in the integration of the least squares pseudo-spectral method with monomial basis functions and Tikhonov regularization for the stable approximation of both linear and nonlinear fractional differential equations. While orthogonal polynomial-based approaches are widely used, the present study demonstrates that, with appropriate regularization, simpler monomial bases can still yield accurate and stable solutions.

The paper is organized as follows: Section 2. gives the preliminaries and definitions. Section 3. describes the least square pseudo-spectral method for linear and non-linear fractional differential equations. Section 4. provides error analysis and estimate for the proposed approximate solution. Section 5. presents some examples and discusses the results. Finally, Section 6. concludes the study.

## 2. PRELIMINARIES AND DEFINITIONS

This section presents some preliminaries and definitions of the fractional calculus theory used in this study.

**Definition 1.** The Riemann-Liouville fractional differential operator is defined as [34]

$${}_0^{RL}D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{-n+\gamma+1}}, \quad \text{for } \gamma \in \mathbb{R}, \quad n-1 \leq \gamma < n, \quad (1)$$

where  $\Gamma$  is the gamma function, which is a generalization of the factorial function, that is

$$\Gamma(n) = n\Gamma(n) = (n-1)! = \int_0^\infty t^{n-1}e^{-t}dt, \quad \text{for } n \in \mathbb{R}_+. \quad (2)$$

**Definition 2.** The Caputo fractional derivative is defined as [34]

$${}_0^CD_t^\gamma f(t) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^n(\tau)d\tau}{(t-\tau)^{-n+\gamma+1}} & \text{for } \gamma \in \mathbb{R}, \quad n-1 \leq \gamma < n, \\ \frac{d^n f(t)}{dt^n} & \text{for } \gamma = n, \quad n \in \mathbb{N}, \end{cases} \quad (3)$$

where  $\Gamma$  is the gamma function.

**Definition 3.** We define the monomial basis as the linear vector space consisting of all monomials  $\{1, t, t^2, t^3, \dots\}$ . [35]

**Lemma 4.** The Caputo derivative of a monomial function has the following form [36]

$${}_0^CD_t^\gamma t^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-\gamma+1)} t^{k-\gamma} & \text{for } k \in \mathbb{N}_0, \quad k \geq \lceil \gamma \rceil, \\ 0, & \text{for } k \in \mathbb{N}_0, \quad k < \lceil \gamma \rceil. \end{cases} \quad (4)$$

**Definition 5.** Let  $N \geq 1$  be an integer, and let  $t_0, t_1, \dots, t_N$  be a set of distinct points in  $[-1, 1]$ . For definiteness, let the numbering be in reverse order, so that  $1 = t_0 > t_1 > \dots > t_{N-1} > t_N = -1$ . The following sets of points are used [37]

(i) Equispaced points:

$$t_j = 1 - \frac{2j}{N}, \quad 0 \leq j \leq N, \quad (5)$$

(ii) *Gauss-Lobatto points:*

$$t_j = \cos \frac{j\pi}{N}, \quad 0 \leq j \leq N, \quad (6)$$

(iii) *Chebyshev points:*

$$t_j = \cos \left( \frac{2j+1}{2N+2} \pi \right) \cos \left( \frac{\pi}{2N+2} \right)^{-1}, \quad 0 \leq j \leq N. \quad (7)$$

The examples presented in this study are bounded over the interval  $[a, b]$ ; therefore, there is a need to transform the interval from the physical domain  $t \in [a, b]$  to the computational domain  $z \in [-1, 1]$  through the following linear mapping

$$t = \frac{1}{2}(b-a)z + \frac{1}{2}(b+a), \quad t \in [a, b]. \quad (8)$$

### 3. THE LEAST SQUARES PSEUDO-SPECTRAL METHOD

The method proposed in this study is premised on the hybridization of the pseudo-spectral method and the least squares method. The pseudo-spectral method, also known as a spectral collocation method, involves approximating the solution of a differential equation as a linear combination of basis functions, mostly orthogonal polynomials, and finding the solution so that it satisfies the differential equation at selected points in the domain called the collocation points [13]. The unknown coefficients in the assumed solution are then evaluated through the least squares method. The least-squares technique involves minimizing the sum of squares of the residual function of the differential equation [38].

In this study, monomial basis functions are employed in constructing the approximate solution. Although monomial basis functions offer simplicity in implementation, they lack orthogonality, which makes the resulting system sensitive to the number of basis functions and contributes to numerical ill-conditioning.

This section describes the proposed method for linear and non-linear fractional differential equations. In the case of non-linear equations, a quasilinearization approach, a technique based on the Newton-Raphson method, is first used to linearize the equations before applying the least square pseudo-spectral method. Further, for non-linear fractional differential equations, the Tikhonov regularization technique is used to address the density of the resulting linearized equations.

#### 3.1. Linear fractional differential equations

Consider the linear fractional differential equation

$$\mathcal{L}[y(t), {}_0^C D_t^\gamma y(t)] = s(t), \quad a < t < b, \quad (9)$$

subject to

$$y(a) = \gamma \quad \text{and} \quad y(b) = \delta, \quad (10)$$

where  $s(t)$  is a known function of  $t$ ,  $\gamma$  and  $\delta$  are known constants and  $\mathcal{L}$  is a linear operator of fractional and/or classical differential operators. Assume that the unknown,  $y(t)$  can be approximated as a linear combination of  $(N+1)$  basis functions,  $\phi(t)$  so that the solution is given as

$$y(t) \approx Y_N(t) = \sum_{i=0}^N h_i \phi_i(t). \quad (11)$$

If Equation (11) is substituted into Equation (9) and  $(N+1)$  collocation points  $t_0, t_1, \dots, t_N$  which lie in the physical domain  $[a, b] \equiv [t_0, t_N]$  are introduced, the residual functions, for  $a < t < b$  at  $t_1, t_2, \dots, t_N$ , are defined as

$$R_j(h_i) = \mathcal{L} \left[ \sum_{i=0}^N h_i \phi_i(t_j), \sum_{i=\lceil \gamma \rceil}^N h_i {}_0^C D_t^\gamma \phi_i(t_j) \right] - s(t_j), \quad j = 1, 2, \dots, N-1. \quad (12)$$

The corresponding residual functions for boundary conditions (10) are defined as follows

$$\left. \begin{aligned} R_0(h_i) &= \sum_{i=0}^N h_i \phi_i(t_0) - \gamma \quad \text{and} \\ R_N(h_i) &= \sum_{i=0}^N h_i \phi_i(t_N) - \delta. \end{aligned} \right\} \quad (13)$$

The least square method is then used to evaluate the coefficients,  $h_0, h_1, \dots, h_N$ , which minimize the sum of squares of the residual functions, that is the minimum of the function

$$q(h_0, h_1, \dots, h_N) = \sum_{j=0}^N \left| R_j(h_i) \right|^2. \quad (14)$$

The minimum of  $q(h_0, h_1, \dots, h_N)$  is calculated by setting  $\frac{\partial q}{\partial h_0}, \frac{\partial q}{\partial h_1}, \dots, \frac{\partial q}{\partial h_N}$  to zero and solving the resulting system of  $(N + 1)$  equations whose unknowns are  $h_0, h_1, \dots, h_N$ .

### 3.2. Non-linear fractional differential equations

To extend the proposed Least Squares Pseudo-Spectral Method (LSPSM) to nonlinear fractional differential equations, an appropriate linearization strategy is required. Since the LSPSM technique is naturally formulated for linear equations, the nonlinear operator is first transformed into a sequence of linear systems. In this study, the quasilinearization technique, based on the generalized Newton-Raphson method, which is based on truncated Taylor's series approximation, is used. At each iteration, the nonlinear equation is approximated by a linear equation, which is then solved using the LSPSM technique.

Consider the non-linear fractional differential equation

$$\mathcal{N}[y(t), y^\gamma(t), y^{(p)}(t)] = s(t), \quad a < t < b, \quad (15)$$

subject to

$$y(a) = \gamma \quad \text{and} \quad y(b) = \delta, \quad (16)$$

where  $y^\gamma(t) \equiv {}^C_0 D_t^\gamma y(t)$ ,  $s(t)$  is a known function of  $t$ ,  $\gamma$  and  $\delta$  are constant,  $\mathcal{N}$  is a non-linear operator of fractional and classical differential operators and  $p$  represent integer order derivatives. Using the generalized Newton-Raphson method, the nonlinear equation (15) is linearized as follows

$$\mathcal{N}_r \left[ y(t), y^\gamma(t), y^{(p)}(t) \right] + \frac{\partial \mathcal{N}}{\partial y_r^\gamma} (y_{r+1}^\gamma - y_r^\gamma) + \sum_{p=0}^P \frac{\partial \mathcal{N}}{\partial y_r^p} (y_{r+1}^p - y_r^p) - s(t) = 0, \quad (17)$$

where  $P$  is the highest integer order derivative. At each iteration step, the resulting linear system is discretized and solved using the proposed LSPSM technique. Assume that the solution of Equation (15) can be approximated as a linear combination of  $(N + 1)$  basis functions,  $\phi(t)$ , then, the following are defined

$$\left\{ \begin{array}{l} y_r(t) \approx Y_r(t) = \sum_{i=0}^N h_{i,r} \phi_i(t), \\ y_{r+1}(t) \approx Y_{r+1}(t) = \sum_{i=0}^N h_{i,r+1} \phi_i(t), \\ y_r^\gamma(t) \approx {}^C_0 D_t^\gamma Y_r(t) = \sum_{i=\lceil \gamma \rceil}^N h_{i,r} {}^C_0 D_t^\gamma \phi_i(t), \\ y_{r+1}^\gamma(t) \approx {}^C_0 D_t^\gamma Y_{r+1}(t) = \sum_{i=\lceil \gamma \rceil}^N h_{i,r+1} {}^C_0 D_t^\gamma \phi_i(t), \\ y_r^p(t) \approx \frac{d^p}{dt^p} Y_r(t) = \sum_{i=p}^P h_{i,r} \frac{d^p}{dt^p} \phi_i(t), \\ y_{r+1}^p(t) \approx \frac{d^p}{dt^p} Y_{r+1}(t) = \sum_{i=p}^P h_{i,r+1} \frac{d^p}{dt^p} \phi_i(t). \end{array} \right. \quad (18)$$

Substituting Equation (18) into Equation (17) and introducing  $(N + 1)$  collocation points  $t_0, t_1, \dots, t_N$  lying in the physical domain  $[a, b] \equiv [t_0, t_N]$  yields the residual function

$$\begin{aligned} R_j(h_i) = & \mathcal{N} \left[ \sum_{i=0}^N h_{i,r} \phi_i(t_j), \sum_{i=\lceil \gamma \rceil}^N h_{i,r} {}^C_0 D_t^\gamma \phi_i(t_j), \sum_{i=p}^N h_{i,r} \frac{d^p}{dt^p} \phi_i(t_j) \right] + \frac{\partial \mathcal{N}}{\partial y_r^\gamma} \sum_{i=\lceil \gamma \rceil}^N h_{i,r+1} {}^C_0 D_t^\gamma \phi_i(t_j) \\ & - \frac{\partial \mathcal{N}}{\partial y_r^\gamma} \sum_{i=\lceil \gamma \rceil}^N h_{i,r} {}^C_0 D_t^\gamma \phi_i(t_j) + \sum_{p=0}^P \frac{\partial \mathcal{N}}{\partial y_r^p} \sum_{i=p}^N h_{i,r+1} \frac{d^p}{dt^p} \phi_i(t_j) - \sum_{p=0}^P \frac{\partial \mathcal{N}}{\partial y_r^p} \sum_{i=p}^N h_{i,r} \frac{d^p}{dt^p} \phi_i(t_j) - s(t_j), \\ & j = 1, 2, \dots, N - 1. \end{aligned} \quad (19)$$

The corresponding boundary conditions in Equation (16) are defined as follows

$$\left\{ \begin{array}{l} R_0(h_i) = \sum_{i=0}^N h_{i+1} \phi_i(t_0) - \gamma \quad \text{and} \\ R_N(h_i) = \sum_{i=0}^N h_{i+1} \phi_i(t_N) - \delta, \end{array} \right. \quad (20)$$

Using the least square method to evaluate the coefficients,  $h_0, h_1, \dots, h_N$  minimizes the sum of squares of the residual function which is defined as

$$q(h_0, h_1, \dots, h_N) = \sum_{j=0}^N \left| R_j(h_i) \right|^2. \quad (21)$$

The minimum of  $q(h_0, h_1, \dots, h_N)$  is obtained by setting  $\frac{\partial q}{\partial h_0}, \frac{\partial q}{\partial h_1}, \dots, \frac{\partial q}{\partial h_N}$  to zero and solving the resulting system of  $(N + 1)$  equations with unknowns  $h_0, h_1, \dots, h_N$ . However, a regularized form of Equation (21) is minimized instead of minimizing Equation (21) directly. The Tikhonov technique is used to regularize the system to become well-posed. To apply the Tikhonov regularization technique, the system is redefined as follows [27]

$$\left[ \text{diag}(A_\gamma) \Phi^\gamma + \sum_{p=0}^P \text{diag}(A_p) \Phi^p \right] h_{r+1} - W, \quad (22)$$

with boundary conditions (20) defined as follows

$$\Phi_0 h_{r+1} - \gamma \quad \text{and} \quad \Phi_N h_{r+1} - \delta, \quad (23)$$

where  $A_\gamma \equiv \frac{\partial \mathcal{N}}{\partial y_r^\gamma}$ ,  $A_p \equiv \frac{\partial \mathcal{N}}{\partial y_r^p}$ ,  $W \equiv s(t_j) + \left[ \text{diag}(A_\gamma) \Phi^\gamma + \sum_{p=0}^P \text{diag}(A_p) \Phi^p \right] C_r - \mathcal{N}$ ,  $\Phi_0 \equiv \phi_i(t_j)$ ,  $\Phi_N \equiv \phi_i(t_N)$ ,  $j = 1, 2, \dots, N-1$ ,  $\Phi^\gamma$  is the matrix whose entries are  ${}_0^C D_t^\gamma \phi_i(t_j)$ ,  $\Phi^p$  is the matrix whose entries are  $\frac{d^p}{dt^p} \phi_i(t_j)$  and  $\Phi^0$  is equivalent to the matrix whose entries are  $\phi_i(t_j)$ . We define the matrix  $B$ , a vertical stacking of

$$\text{diag}(A_\gamma) \Phi^\gamma + \sum_{p=0}^P \text{diag}(A_p) \Phi^p, \quad \phi_0 \quad \text{and} \quad \phi_N.$$

Then, the Tikhonov regularization minimizes the following

$$\|B H_{r+1} - W\|^2 + \sigma^2 \|H_{r+1}\|^2, \quad (24)$$

where  $\sigma$  is the regularization parameter. The residual defined by Equation (24) means that we seek the least square solution of

$$\begin{bmatrix} B \\ \sigma I \end{bmatrix} H_{r+1} = \begin{bmatrix} W \\ 0 \end{bmatrix}. \quad (25)$$

The pseudo-inverse solution of the system (25) is given as

$$[B^T B + \sigma I] H_{r+1} = [B^T \quad \sigma I] \begin{bmatrix} W \\ 0 \end{bmatrix}, \quad (26)$$

where  $B^T$  is the transposed of matrix  $B$ , and so the solution of the system (26) is given as

$$H_{r+1} = [B^T B + \sigma I]^{-1} B^T W. \quad (27)$$

It can be seen that as  $\sigma \rightarrow 0$ , the effect of regularization disappears and the system reverts to its non-regularized form.

#### 4. ERROR ANALYSIS AND ESTIMATE

This section presents error analysis and bounds for the proposed approximate solution. We assume that the proposed solution (11),  $Y_N(t)$ , to the fractional differential equation is sufficiently smooth on  $[a, b]$ .

##### 4.1. Residual error

Consider the linear fractional differential equation (9) with associated boundary conditions (10). An approximate solution of the form (11) is considered so that the residual functions are given as

$$\left. \begin{aligned} R_j(h_i) &= \mathcal{L} \left[ \sum_{i=0}^N h_i \phi_i(t_j), \sum_{i=\lceil \gamma \rceil}^N h_i {}_a^C D_t^\gamma \phi_i(t_j) \right] - s(t_j), \quad j = 1, 2, \dots, N-1, \\ R_0(h_i) &= \sum_{i=0}^N h_i \phi_i(t_0) - \gamma, \\ R_N(h_i) &= \sum_{i=0}^N h_i \phi_i(t_N) - \delta. \end{aligned} \right\} \quad (28)$$

**Remark 6.** The set of equations (28) provides a total of  $N+1$  algebraic equations. These equations correspond to  $N+1$  unknowns, namely  $h_0, h_1, h_2, \dots, h_N$ , which are determined using the least square method.

**Remark 7.** For non-linear fractional differential equations, a corresponding system of  $N + 1$  algebraic equations is derived with  $N + 1$  unknowns. The residual functions are defined by Equations (19) and (20).

Consider  $y(t)$  represents the exact solution of the fractional differential equation, and  $Y_N(t)$  denotes the approximate solution. The error can then be expressed as

$$\zeta_N(t) = |y(t) - Y_N(t)|. \quad (29)$$

Additionally, the residual error can also be used to check the accuracy of solutions obtained using the numerical method. If we have determined the solution  $Y_N(t)$  along with its derivatives, then the solution must satisfy

$$\begin{cases} R_j = \mathcal{L} [Y_N(t_j), {}^C D_t^\gamma Y_N(t_j)] - s(t_j) \approx 0, & \text{and} \\ R_j = \mathcal{N} \left[ Y_N(t_j), {}^C D_t^\gamma Y_N(t_j), \frac{d^{(p)}}{dt^{(p)}} Y_N(t_j) \right] - s(t_j) \approx 0, & j = 0, 1, \dots, N, \end{cases} \quad (30)$$

for linear and non-linear fractional differential equations, respectively.

#### 4.2. Error bounds

This section proceed to estimate the upper bound for the best approximation and the derivatives of the best approximation, classical and fractional, alike.

**Lemma 8.** Suppose  $\frac{d^i y(t)}{dt^i} \in C[a, b]$ , for  $i = 0, 1, \dots, N$ , by Taylor's approximation [39]

$$y(t) = \sum_{i=0}^N h_i (t-a)^i, \quad \text{with} \quad h_i = \frac{1}{\Gamma(i+1)} \frac{d^i y(a)}{dt^i}, \quad (31)$$

so that

$$y(t) = \sum_{i=0}^N h_i (t-a)^i + \frac{(t-a)^{N+1}}{\Gamma(N+2)} \frac{d^{N+1} y(\tau)}{dt^{N+1}}, \quad (32)$$

with  $a < \tau \leq t$ , for all  $t \in [a, b]$ . Therefore,

$$\left| y(t) - \sum_{i=0}^N h_i (t-a)^i \right| = |y(t) - Y_N(t)| \leq \frac{M(t-a)^{N+1}}{\Gamma(N+2)}, \quad (33)$$

where  $\left| \frac{d^{N+1} y(\tau)}{dt^{N+1}} \right| \leq M$ ,  $M \in \mathbb{R}$ .

**Theorem 9** (Error bound for the best approximation). Suppose  $\frac{d^i y(t)}{dt^i} \in C[a, b]$ , for  $i = 0, 1, \dots, N$ ,  $t \in [a, b]$  and  $\Phi_N = \text{span}\{1, t-a, (t-a)^2, \dots\}$ . Let  $y(t)$ , a real-valued sufficiently smooth function, be the exact solution of a fractional differential equation, and  $Y_N(t) = \sum_{i=0}^N h_i \phi_i(t)$  be the best approximation of  $y(t)$  from  $\Phi_N$ , then the upper bound of the error is estimated as

$$\|y(t) - Y_N(t)\|_2 \leq \frac{M(b-a)^{N+\frac{3}{2}}}{\Gamma(N+2)\sqrt{2N+3}}, \quad (34)$$

where  $M = \max_{t \in [a, b]} \left| \frac{d^{N+1} y(t)}{dt^{N+1}} \right|$ .

*Proof.* Since  $Y_N(t)$  is the best approximation of  $y(t)$ , and by use of Theorem 8,

$$\|y(t) - Y_N(t)\|_2 \leq \frac{M}{\Gamma(N+2)} \left( \int_a^b ((t-a)^{N+1})^2 dt \right)^{\frac{1}{2}} = \frac{M(b-a)^{N+\frac{3}{2}}}{\Gamma(N+2)\sqrt{2N+3}}, \quad (35)$$

as required. □

**Theorem 10** (Upper bound for the  $n$ -th order classical differentiation). *Suppose that the conditions in Theorem 9 are satisfied, then*

$$\left\| \frac{d^n}{dt^n} y(t) - \frac{d^n}{dt^n} Y_N(t) \right\|_2 \leq \frac{M(b-a)^{N-n+\frac{3}{2}}}{\Gamma(N-n+2)\sqrt{2N-2n+3}}, \quad (36)$$

where  $n = \lceil \gamma \rceil$ .

*Proof.* Using Theorem 8, we have the following:

$$\left. \begin{aligned} \left\| \frac{d^n}{dt^n} y(t) - \frac{d^n}{dt^n} Y_N(t) \right\|_2 &= \left\| \frac{d^n}{dt^n} (y(t) - Y_N(t)) \right\|_2 \leq \left\| \frac{d^n}{dt^n} \frac{M(t-a)^{N+1}}{\Gamma(N+2)} \right\|_2 \\ &= \left\| \frac{M(t-a)^{N-n+1}}{\Gamma(N-n+2)} \right\|_2 = \frac{M(b-a)^{N-n+\frac{3}{2}}}{\Gamma(N-n+2)\sqrt{2N-2n+3}}, \end{aligned} \right\} \quad (37)$$

which is the required result. □

**Theorem 11** (Error bound for fractional differentiation). *Let the conditions specified in Theorem 9 be satisfied for  $n-1 \leq \gamma < n$ , where  $n = \lceil \gamma \rceil$ . Then,*

$$\| {}^C_a D_t^\gamma y(t) - {}^C_a D_t^\gamma Y_N(t) \|_2 \leq \frac{M(b-a)^{N-\gamma+\frac{3}{2}}}{\Gamma(N-\gamma+2)\sqrt{2N-2\gamma+3}}. \quad (38)$$

*Proof.* By using the Caputo fractional integral formula and Theorem 8, we have the following

$$\left. \begin{aligned} \| {}^C_a D_t^\gamma y(t) - {}^C_a D_t^\gamma Y_N(t) \|_2 &\leq \left\| \frac{M}{\Gamma(n-\gamma)\Gamma(N-n+2)} \int_a^t (\tau-a)^{N-n+1} (t-\tau)^{n-\gamma-1} d\tau \right\|_2 \\ &= \left\| \frac{M(t-a)^{N-\gamma+1}}{\Gamma(N-\gamma+2)} \right\|_2 = \frac{M(b-a)^{N-\gamma+\frac{3}{2}}}{\Gamma(N-\gamma+2)\sqrt{2N-2\gamma+3}}, \end{aligned} \right\} \quad (39)$$

which completes the proof. □

## 5. NUMERICAL EXAMPLES

This section presents examples to illustrate the application of the proposed numerical technique on linear and non-linear fractional differential equations.

**Example 5.1..** Consider the linear fractional differential equation

$${}_0^C D_t^\gamma y(t) + \beta y(t) = g(t), \quad 1 < \gamma \leq 2, \quad (40)$$

subject to boundary conditions

$$y(0) = 0, \quad y(1) = \xi y(\eta), \quad (41)$$

and

$$g(t) = \frac{\Gamma(6)}{\Gamma(6-\gamma)} t^{5-\gamma} - \frac{37}{20} \frac{\Gamma(4)}{\Gamma(4-\gamma)} t^{3-\gamma} + \frac{33}{40} \frac{\Gamma(3)}{\Gamma(3-\gamma)} t^{2-\gamma} + \beta(t^5 - \frac{37}{20} t^3 + \frac{33}{40} t^2),$$

for

$$\gamma = \frac{3}{2}, \quad \beta = \frac{e^{3\pi}}{\sqrt{3}}, \quad \xi = -\frac{125}{196}, \quad \eta = \frac{2}{5}, \quad y(1) = -\frac{1}{40},$$

with exact solution

$$y(t) = \left( t^3 - \frac{37}{20}t + \frac{33}{40} \right) t^2. \quad (42)$$

The residual function is obtained as

$$R_j(h_i) = \sum_{i=2}^N h_i \frac{\Gamma(i+1)}{\Gamma(i-\gamma+1)} t_j^{i-\gamma} + \beta \sum_{i=0}^N h_i t_j^i - g(t_j), \quad j = 1, 2, \dots, N-1, \quad (43)$$

with residuals for the boundary conditions defined as

$$R_0(h_i) = h_0 \quad \text{and} \quad (44)$$

$$R_N(h_i) = \sum_{i=0}^N h_i - 1. \quad (45)$$

Table 1. Computational time in seconds for Example 5.1. for different values of N with  $\gamma = 1.5$ .

| N   | Equispaced | Chebyshev | Gauss-Lobatto |
|-----|------------|-----------|---------------|
| 15  | 0.000038   | 0.000030  | 0.000034      |
| 30  | 0.000044   | 0.000040  | 0.000035      |
| 50  | 0.000053   | 0.000042  | 0.000041      |
| 100 | 0.000060   | 0.000048  | 0.000043      |

Table 2. Computational time in seconds for Example 5.1. for different values of  $\gamma$  with N = 50

| $\gamma$ | Equispaced | Chebyshev | Gauss-Lobatto |
|----------|------------|-----------|---------------|
| 1        | 0.000032   | 0.000045  | 0.000031      |
| 1.5      | 0.000035   | 0.000033  | 0.000047      |
| 1.75     | 0.000041   | 0.000032  | 0.000058      |
| 2        | 0.000052   | 0.000003  | 0.000134      |

Table 3. Condition number of the system matrix for Example 5.1. for different values of N with  $\gamma = 1.5$ .

| N   | Equispaced | Chebyshev  | Gauss-Lobatto |
|-----|------------|------------|---------------|
| 15  | 1.8140e+11 | 1.1738e+10 | 8.5975e+09    |
| 30  | 1.9068e+19 | 8.2348e+18 | 3.3218e+19    |
| 50  | 2.3429e+19 | 2.1888e+21 | 2.9826e+20    |
| 100 | 1.0747e+21 | 3.1146e+21 | 3.6805e+21    |

Tables 1 and 2 present the computational time in seconds for Example 5.1. for different values of N and  $\gamma$ , respectively for equispaced, Chebyshev and Gauss-Lobatto collocation points. In Table 1, it is shown that the Gauss-Lobatto collocation points performs better for different values of N. For different values of  $\gamma$ , from Table 2 it is evident that Chebyshev collocation points performs better. It is clearly seen from Table 2 that the least square pseudo-spectral method works much better with the fractional power in Example 5.1. when using equispaced and Gauss-Lobatto collocation points. The least square pseudo-spectral method proved to have a high convergence rate.

Table 3 presents the condition numbers of the resulting system matrices for different values of  $N$  and various collocation strategies. It is observed that the condition number increases rapidly with  $N$ , reaching values as high as  $10^{21}$ , indicating severe ill-conditioning. This behavior is consistent with the known limitations of monomial basis functions, which tend to produce poorly conditioned systems for high-degree approximations. Although using Chebyshev and Gauss-Lobatto collocation points slightly improves conditioning compared to equispaced points, the overall system remains highly sensitive to perturbations. To address this issue, Tikhonov regularization is employed to solve the resulting algebraic systems, particularly for nonlinear problems, where stability is more critical. Despite the large condition numbers, the numerical results demonstrate that the proposed method remains accurate and stable when appropriate regularization is applied.

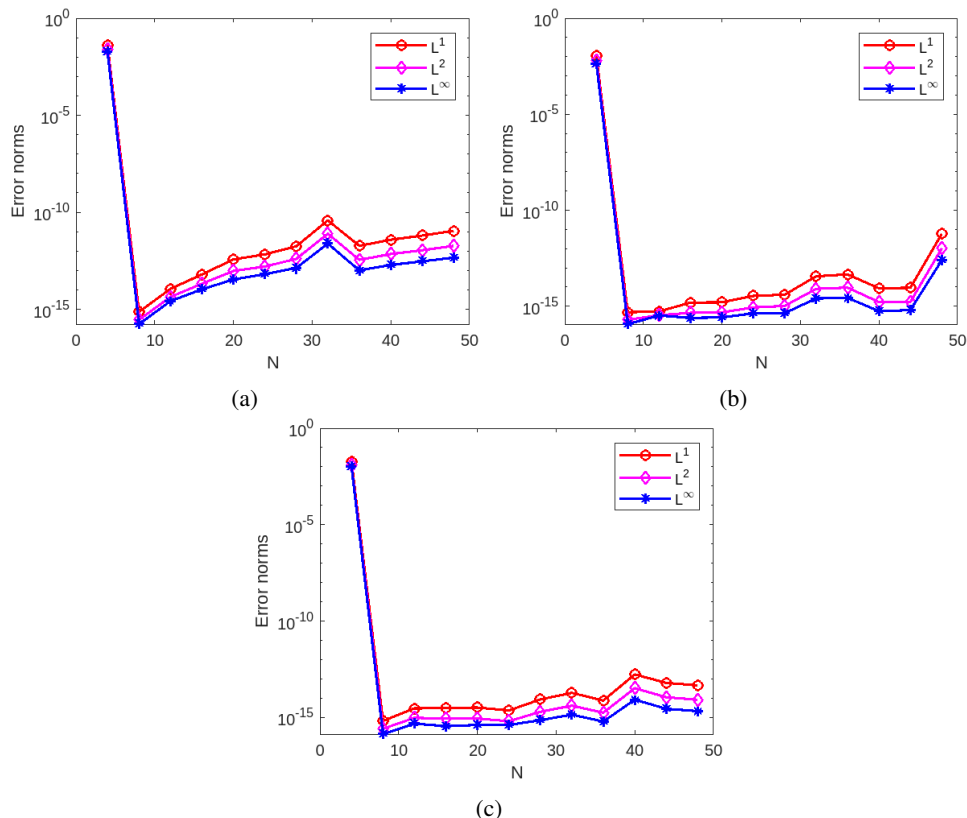


Figure 1.  $L^1$ ,  $L^2$  and  $L^\infty$  norms of  $|y(t) - Y_N(t)|$  for Example 5.1. with  $\gamma = 1.5$  using (a) Equispaced (b) Gauss-Lobatto and (c) Chebyshev collocation points.

The performance of the proposed method is further assessed using the  $L_1$ ,  $L_2$ , and  $L_\infty$  error norms. These norms provide a comprehensive measure of the deviation between the exact and numerical solutions over the computational domain. The figures show that all error norms decrease as the number of basis functions increases, confirming the convergence and stability of the proposed LSPSM technique. Figure 1 presents the  $L^1$ ,  $L^2$  and  $L^\infty$  norms of  $|y(t) - Y_N(t)|$  for Example 5.1. with  $\gamma = 1.5$ . The results show that the numerical solutions are in agreement with the exact solutions. The figure shows that the least square pseudo-spectral method gives more accurate solution when the Chebyshev collocation points are used compared to the equispaced and Gauss-Lobatto collocation points. The method proves to be reliable, effective and accurate for this illustrative example.

**Example 5.2..** Consider the linear fractional differential equation:

$${}_0^C D_t^\gamma y(t) + \lambda {}_0^C D_t^\beta y(t) + \nu y(t) = g(t), \quad t \in [0, 1], \quad 1 < \gamma \leq 2, \quad 0 < \beta \leq 1, \quad (46)$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0, \quad (47)$$

and

$$g(t) = p(t) + \lambda q(t) + \nu r(t),$$

where

$$p(t) = 625 \frac{\Gamma(8)}{\Gamma(8-\gamma)} t^{7-\gamma} - 2000 \frac{\Gamma(7)}{\Gamma(7-\gamma)} t^{6-\gamma} + 2450 \frac{\Gamma(6)}{\Gamma(6-\gamma)} t^{5-\gamma} - 1420 \frac{\Gamma(5)}{\Gamma(5-\gamma)} t^{4-\gamma} \\ + 318 \frac{\Gamma(4)}{\Gamma(4-\gamma)} t^{3-\gamma} - 36 \frac{\Gamma(3)}{\Gamma(3-\gamma)} t^{2-\gamma},$$

$$q(t) = 625 \frac{\Gamma(8)}{\Gamma(8-\beta)} t^{7-\beta} - 2000 \frac{\Gamma(7)}{\Gamma(7-\beta)} t^{6-\beta} + 2450 \frac{\Gamma(6)}{\Gamma(6-\beta)} t^{5-\beta} - 1420 \frac{\Gamma(5)}{\Gamma(5-\beta)} t^{4-\beta} \\ + 318 \frac{\Gamma(4)}{\Gamma(4-\beta)} t^{3-\beta} - 36 \frac{\Gamma(3)}{\Gamma(3-\beta)} t^{2-\beta},$$

and

$$r(t) = 625t^7 - 2000t^6 + 2450t^5 - 1420t^4 + 381t^3 - 36t^2$$

for

$$\gamma = \frac{7}{4}, \quad \beta = \frac{1}{2}, \quad \lambda = 1, \quad \rho = -\frac{1}{\sqrt{\pi}}.$$

The exact solution of Equation (46) subject to the boundary conditions is given as

$$y(t) = 625t^7 - 2000t^6 + 2450t^5 - 1420t^4 + 381t^3 - 36t^2. \quad (48)$$

The residual function is obtained as

$$R_j(h_i) = \sum_{i=2}^N h_i \frac{\Gamma(i+1)}{\Gamma(i-\gamma+1)} t_j^{i-\gamma+\lambda} + \sum_{i=1}^N h_i \frac{\Gamma(i+1)}{\Gamma(i-\beta+1)} t_j^{i-\beta+\nu} + \sum_{i=0}^N h_i t_j^i - g(t_j), \quad j = 1, 2, \dots, N-1,$$

with residuals for the boundary conditions defined as

$$R_0(h_i) = h_0 \quad \text{and} \quad (49)$$

$$R_N(h_i) = \sum_{i=0}^N h_i - 1. \quad (50)$$

Table 4. Computational time in seconds for Example 5.2. for different values of  $N$  with  $\gamma = 1.75$ .

| N   | Equispaced | Chebyshev | Gauss-Lobatto |
|-----|------------|-----------|---------------|
| 15  | 0.0000306  | 0.000038  | 0.000034      |
| 30  | 0.000032   | 0.000031  | 0.000031      |
| 50  | 0.000032   | 0.000030  | 0.000029      |
| 100 | 0.000030   | 0.000002  | 0.000003      |

The computational time in seconds for Example 5.2. for different values of  $N$  and  $\gamma$  are presented in Tables 4 and 5, respectively. In Tables 4 and 5 it is clearly seen that the least square pseudo-spectral method is computationally efficient when the Chebyshev and Gauss-Lobatto collocation points are used in comparison to the equispaced collocation points.

In Figure 2 the  $L^1$ ,  $L^2$  and  $L^\infty$  norms of  $|y(t) - Y_N(t)|$  for Example 5.2. for  $\gamma = 1.75$ . are presented for equispaced, Gauss-Lobatto and Chebyshev collocation points. The results shows that the numerical solutions are in agreement with the exact solutions with the Gauss-Lobatto collocation points giving more accurate

Table 5. Computational time in seconds for Example 5.2. for different values of  $\gamma$  with  $N = 15$ 

| $\gamma$ | Equispaced | Chebyshev | Gauss-Lobatto |
|----------|------------|-----------|---------------|
| 1        | 0.000032   | 0.000033  | 0.000033      |
| 1.5      | 0.000047   | 0.000040  | 0.000036      |
| 1.75     | 0.000061   | 0.000057  | 0.000043      |
| 2        | 0.000036   | 0.000033  | 0.000044      |

results. The results presented in Figure 2 shows that the method is effective and accurate for solving fractional ordinary differential equations.

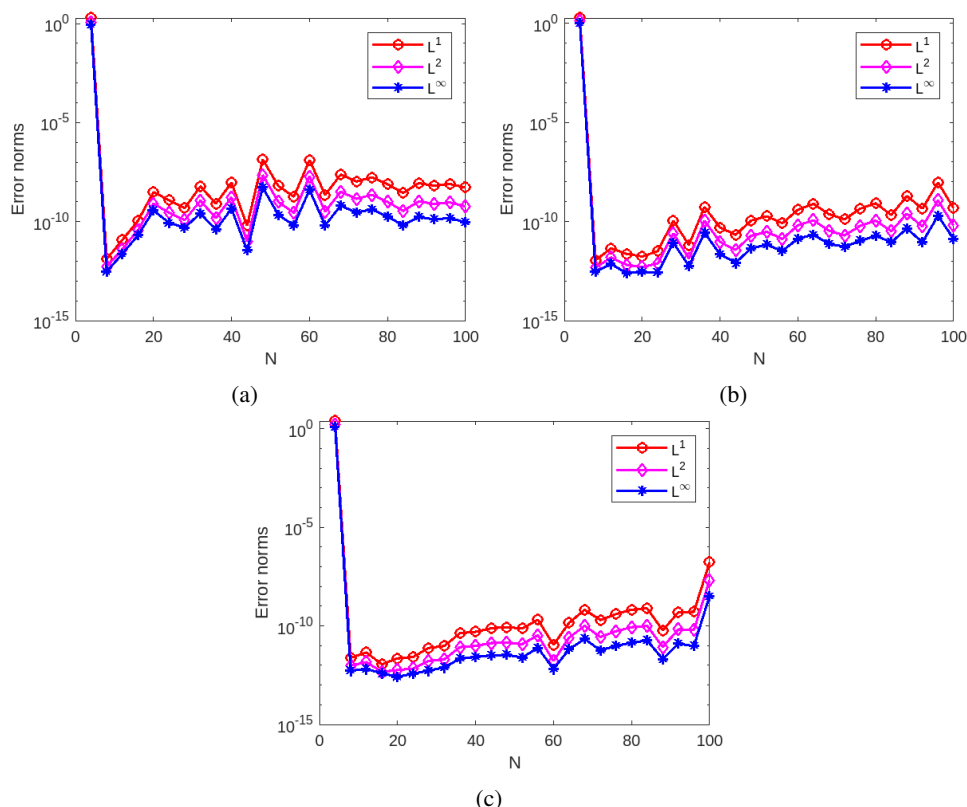


Figure 2.  $L^1$ ,  $L^2$  and  $L^\infty$  norms of  $|y(t) - Y_N(t)|$  for Example 5.2. for  $\gamma = 1.75$ . using (a) Equispaced (b) Gauss-Lobatto and (c) Chebyshev collocation points.

**Example 5.3..** Consider the non-linear fractional differential equation

$${}_0^C D_t^\gamma y(t) + ty^2(t) = 2 + t^5, \quad 1 < \gamma \leq 2 \quad (51)$$

subjected to

$$y(0) = 0, \quad y(1) = 1, \quad (52)$$

with exact solution

$$y(t) = t^2.$$

The residual function for the non-linear Equation (51) after linearization is given as

$$R_j(h_i) = \sum_{i=1}^N h_{i,r+1} \frac{\Gamma(i+1)}{\Gamma(i-\gamma+1)} t_j^{i-\gamma} + 2t_j y_r \sum_{i=0}^N h_{i,r+1} t_j^i - W, \quad j = 1, 2, 3, \dots, N-1,$$

using the boundary conditions (52), the initial guess for  $H$  is obtained as

$$y_r = t.$$

Table 6. Residual norm ( $\|R_j(h_i)\|_2$ ) for regularized and non-regularized schemes of Example 5.3. for different values of  $N$  with  $\gamma = 1.75$ .

| N   | Equispaced<br>$\ R_j(h_i)\ _2$ |            | Gauss-Lobatto<br>$\ R_j(h_i)\ _2$ |            | Chebyshev<br>$\ R_j(h_i)\ _2$ |            |
|-----|--------------------------------|------------|-----------------------------------|------------|-------------------------------|------------|
|     | Reg                            | Non-reg    | Reg                               | Non-reg    | Reg                           | Non-reg    |
| 2   | 4.7275e-33                     | 0          | 4.7275e-33                        | 1.9722e-31 | 4.7275e-33                    | 1.9722e-31 |
| 4   | 7.7673e-29                     | 2.0954e-31 | 5.6088e-32                        | 1.0845e-30 | 6.6329e-29                    | 2.0117e-31 |
| 8   | 4.3812e-23                     | 5.3694e-28 | 9.0684e-25                        | 5.3279e-27 | 3.1261e-26                    | 1.4467e-27 |
| 16  | 3.4994e-23                     | 9.4863e-20 | 4.9163e-21                        | 3.5076e16  | 4.8822e-22                    | 1.2348e-17 |
| 32  | 1.4105e-18                     | 6.0201e-09 | 4.1499e-19                        | 0.6909     | 3.8309e-18                    | 0.0229     |
| 64  | 1.2135e-14                     | 2.1661e-05 | 2.1965e-12                        | 1.9608e+16 | 9.2212e-13                    | 2.3155e+06 |
| 128 | 5.3332e-11                     | 0.0341     | 4.6907e-08                        | 5.3954e+34 | 2.4077e-08                    | 1.0080e+04 |

Table 6 shows the residual norm  $\|R_j(h_i)\|_2$  for Example 5.3. for different values of  $N$  with  $\gamma = 1.75$ . For non-linear fractional examples, the numerical schemes are regularized; hence, Table 6 compares the residual norms of the regularized and non-regularized schemes. According to the results, regularizing the scheme gives well-posed schemes. As  $N$  increases, the residual norm of the non-regularized scheme proves its ill-posedness, favouring the regularization approach. The residual norm of the regularized scheme proves the accuracy and effectiveness of the proposed method.

**Example 5.4..** Consider the non-linear fractional differential equation,

$${}_0^C D_t^\gamma y(t) + e^t y^3(t) = e^t + e^{4t}, \quad 1 < \gamma \leq 2 \quad (53)$$

subjected to boundary conditions

$$y(0) = 1, \quad y(1) = e. \quad (54)$$

The exact solution for Equation (53) is given as

$$y(t) = e^t.$$

Setting  $D^\gamma y(t) = y^\gamma$  we have

$$y_{r+1}^\gamma + 3e^t y_r^2 y_{r+1} = e^t + e^{4t} + 2e^t y_r^3,$$

using the monomial basis function and introducing the collocation points, the residual function will be defined as

$$R_j(h_i) = \sum_{i=1}^N h_{i,r+1} \frac{\Gamma(i+1)}{\Gamma(i-\gamma+1)} t_j^{i-\gamma} + 3e^t - j y_r^2 \sum_{i=0}^N h_{i,r+1} t_j^i - W, \quad j = 1, 2, 3, \dots, N-1,$$

using the boundary conditions (54), the initial guess for  $H$  is obtained as

$$y_r = (e-1)t + 1.$$

Table 7. Computational time and residual norm ( $\|R_j(h_i)\|_2$ ) of Example 5.4. for different values of  $N$  with  $\gamma = 1.25$ .

| N   | CPU time | Equispaced<br>( $\ R_j(h_i)\ _2$ ) |            | CPU time | Gauss-Lobatto<br>( $\ R_j(h_i)\ _2$ ) |            | CPU time | Chebyshev<br>( $\ R_j(h_i)\ _2$ ) |            |
|-----|----------|------------------------------------|------------|----------|---------------------------------------|------------|----------|-----------------------------------|------------|
|     |          | Reg                                | Non-reg    |          | Reg                                   | Non-reg    |          | Reg                               | Non-reg    |
| 25  | 0.000026 | 1.1030e-13                         | 12.3089    | 0.000028 | 1.0345e-15                            | 4.4153e-04 | 0.000045 | 2.7374e-16                        | 1.1624e-06 |
| 50  | 0.000026 | 4.1709e-13                         | 0.0014     | 0.000028 | 2.2597e-13                            | 1.0993e-05 | 0.000037 | 1.9618e-13                        | 1.3688e-06 |
| 75  | 0.000027 | 1.1054e-13                         | 1.2977e-08 | 0.000028 | 1.0115e-11                            | 1.6554e-05 | 0.000042 | 7.4102e-12                        | 0.0037     |
| 100 | 0.000031 | 4.7041e-12                         | 3.9885e-08 | 0.000047 | 2.2444e-10                            | 2.4887e-04 | 0.000037 | 1.1038e-10                        | 1.4848e-04 |

The computational time in seconds and the residual norm ( $\|R_j(h_i)\|_2$ ) for Example 5.4. are depicted in Table 7 for different values of  $N$  with  $\gamma = 1.25$ . The results show that the least square pseudo-spectral scheme is computationally efficient, as evident in the CPU time. The method proves to be highly accurate

when the equispaced collocation points are used compared to the Gauss-Lobatto and the Chebyshev collocation points. Comparing the residual norms of the regularized and non-regularized schemes, it is evident that the regularized schemes give a more accurate approximation of the exact solution than the non-regularized schemes. The method performs better with fewer collocation points, as evident by the regularized scheme's residual norm.

**Example 5.5..** Consider the following non-linear fractional differential equation,

$${}_0^C D_t^\gamma y(t) + y^3(t)e^y = \frac{\Gamma(\gamma+3)}{2}t^2 + e^{t(\gamma+2)}t^{3(\gamma+2)} \quad 1 < \gamma \leq 2 \quad (55)$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 1, \quad (56)$$

with exact solution

$$y(t) = t^{\gamma+2}.$$

Setting  ${}_0^C D_t^\gamma y(t) = y^\gamma$  we have

$$y_{r+1}^\gamma + (3y_r^2 e^{y_r} + y_r^3 e^{y_r})y_{r+1} = \frac{\Gamma(\gamma+3)}{2}t^2 + e^{t^{\gamma+2}}t^{3(\gamma+2)} + 2y_r^3 e^{y_r} + y_r^4 e^{y_r},$$

using the monomial basis function and introducing the collocation points, the residual function is obtained as

$$R_j(h_i) = \sum_{i=1}^N h_{i,r+1} \frac{\Gamma(i+1)}{\Gamma(i-\gamma+1)} t_j^{i-\gamma} + (3y_r^2 e^{y_r} + y_r^3 e^{y_r}) \sum_{i=0}^N h_{i,r+1} t_j^i - W,$$

using the boundary conditions (56), the initial guess for  $H$  is obtained as

$$y_r = t.$$

Table 8. Computational time, absolute error ( $|y(t) - Y_N(t)|$ ) and residual norm ( $\|R_j(h_i)\|_2$ ) of Example 5.5. for different values of  $N$  with  $\gamma = 1.35$ .

| N   | Equispaced     |                   |                  | Gauss-Lobatto  |                   |                  | Chebyshev      |                   |                  |
|-----|----------------|-------------------|------------------|----------------|-------------------|------------------|----------------|-------------------|------------------|
|     | CPU time(secs) | $ y(t) - Y_N(t) $ | $\ R_j(h_i)\ _2$ | CPU time(secs) | $ y(t) - Y_N(t) $ | $\ R_j(h_i)\ _2$ | CPU time(secs) | $ y(t) - Y_N(t) $ | $\ R_j(h_i)\ _2$ |
| 25  | 0.000023       | 7.7716e-16        | 3.2247e-24       | 0.000022       | 4.4409e-16        | 1.6867e-23       | 0.000038       | 0.000000          | 1.4876e-23       |
| 50  | 0.000023       | 1.4655e-14        | 1.2579e-20       | 0.000040       | 1.8208e-14        | 2.0062e-19       | 0.000051       | 5.1559e-13        | 2.4822e-19       |
| 75  | 0.000021       | 3.9879e-13        | 3.4567e-19       | 0.000024       | 1.3567e-13        | 4.8566e-19       | 0.000022       | 8.4155e-14        | 7.5306e-19       |
| 100 | 0.000020       | 3.3804e-12        | 3.1854e-17       | 0.000023       | 6.2794e-13        | 7.4435e-17       | 0.000034       | 4.3521e-14        | 4.2187e-17       |

For Example 5.5., the computational time in seconds,  $|y(t) - Y_N(t)|$  and  $\|R_j(h_i)\|_2$  for different values of  $N$  with  $\gamma = 1.35$  are presented in Table 8. The results show that the least square pseudo-spectral method is computationally efficient, in terms of CPU time, with equispaced points being the most efficient. The results depicted in Table 8 show that the regularized numerical solution agrees with the exact solution for equispaced, Gauss-Lobatto and Chebyshev collocation points with Chebyshev collocation points giving the best approximation. The method is accurate and effective, evidenced by the magnitude of the residual norm ( $\|R_j(h_i)\|_2$ ).

## 6. CONCLUSION

This study proposed the least square pseudo-spectral method to solve linear and non-linear fractional ordinary differential equations, collocating them using the monomial basis functions. We noticed that the numerical solutions agreed with the exact solution for linear fractional ordinary differential equations, evidenced by the error norms,  $L^1$ ,  $L^2$  and  $L^\infty$ . The method proved computationally efficient, as evidenced by the computational time. The results also confirmed that the numerical approach presented is accurate, effective and reliable for solving linear fractional ordinary differential equations.

Non-linear fractional differential equations were linearized using the quasilinearization approach. The resulting numerical schemes were regularized using the Tikhonov regularization technique. Regularizing numerical schemes leads to well-posed schemes, according to the results. It was shown that the proposed regularized method is more accurate and effective than non-regularized schemes. There is good agreement between the numerical and exact solutions for regularized schemes. The method may be extended to solving fractional partial differential equations in the future.

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